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ON A GENERALIZATION OF
A RESULT OF WINTNER

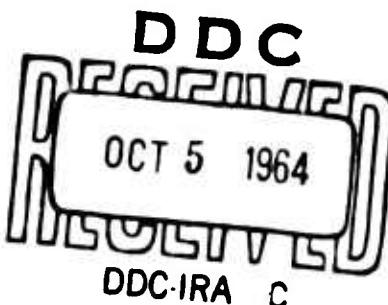
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SUMMARY

We obtain a generalization of the Hukuwara stability theorem analogous to a recent generalization for second order equations due to Wintner, Quarterly of Applied Mathematics, Vol. XV(1958), pp. 428-430.

ON A GENERALIZATION OF A RESULT OF WINTNER

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In a recent note, [1], Wintner proved the following interesting result.

Theorem 1. Consider the two equations

$$(a) \quad u'' + f(t)u = 0, \quad (b) \quad v'' + g(t)v = 0. \quad (1.1)$$

If there exist two linearly independent solutions of (1a), u_1 and u_2 , such that

$$\int^{\infty} (|u_1|^2 + |u_2|^2) |f - g| dt < \infty \quad (1.2)$$

then every solution of (1b) can be written in the form

$$v = c_1 u_1 + c_2 u_2 + O(|u_1| + |u_2|). \quad (1.3)$$

This is an extension of known stability results, cf. [2], to which it reduces if we assume that all solutions of (1a) are bounded as $t \rightarrow \infty$.

Let us now show that we can obtain a generalization of this result following the method used in our book, [2], to establish the Hukuhara stability theorem, of which this will be an extension.

Theorem 2. Consider the vector-matrix systems

$$(a) \quad \frac{dx}{dt} = A(t)x, \quad (b) \quad \frac{dy}{dt} = B(t)y. \quad (1.4)$$

Let $X(t)$ be the solution of

$$\frac{dx}{dt} = A(t)x, \quad x(0) = I. \quad (1.5)$$

If

$$\int^{\infty} ||B(t) - A(t)|| \cdot ||x(t)|| \cdot ||x^{-1}(t)|| dt < \infty, \quad (1.6)$$

then every solution of (1b) may be written

$$y = x_c + O(||x||) \quad (1.7)$$

as $t \rightarrow \infty$.

The norms of matrices and vectors are taken to be respectively $\sum_{i,j} |x_{ij}|$ and $\sum_i |x_i|$.

Proof. Write

$$\frac{dy}{dt} = A(t)y + (B(t) - A(t))y. \quad (1.8)$$

Then, if $y(0) = b$, we have

$$y = x(t)b + \int_0^t x(t)x^{-1}(s)(B(s) - A(s))y ds. \quad (1.9)$$

Hence

$$\begin{aligned} ||y|| &\leq ||x(t)|| \cdot ||b|| + \int_0^t ||x(t)|| \cdot ||x^{-1}(s)|| \cdot ||B(s) \\ &\quad - A(s)|| \cdot ||y|| ds. \end{aligned} \quad (1.10)$$

Thus, if we set

$$u(t) = ||x^{-1}(t)|| \cdot ||B(t) - A(t)|| \cdot ||y(t)||, \quad (1.11)$$

$$v(t) = ||B(t) - A(t)|| \cdot ||x(t)|| \cdot ||x^{-1}(t)||,$$

we obtain the scalar inequality

$$u \leq c_1 v + v \int_0^t u ds. \quad (1.12)$$

This yields, as a consequence of the fundamental inequality, [2], or directly, the estimate

$$\int_0^t u ds \leq c_1 \int_0^t v(s) e^{\int_s^t v dr} ds. \quad (1.13)$$

By assumption, $\int_0^\infty v ds < \infty$. Hence, the integral

$$\int_0^\infty x^{-1}(s)(B(s) - A(s))y ds \quad (1.14)$$

converges. This means that we can write (1.9) in the form

$$y = x(t)b + x(t) \int_0^\infty x^{-1}(s)(B(s) - A(s))y ds \quad (1.15)$$

$$- x(t) \int_t^\infty x^{-1}(s)(B(s) - A(s))y ds,$$

which yields the stated result.

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REFERENCES

1. Wintner, A., "On Linear Perturbations," Quart. Appl. Math., Vol. XV(1958), pp. 428-430.
2. Bellman, R., Stability Theory of Differential Equations, McGraw-Hill Book Company, Inc., New York, 1954.